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# Quasi-exactly-solvable models from finite-dimensional matrices

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**Abstract.** A new method of obtaining many-dimensional quasi-exactly-solvable models is suggested. It is based on constructing the generating function with the help of coefficients which obey a finite difference equation. The structure of this equation is selected to obtain the closed second-order differential equation for the generating function. Under some conditions this equation can be thought of as the Schrödinger equation in curved space. For the two-dimensional case the many-parametric class of solution is found explicitly. The spherically-symmetrical case is investigated in detail. It is shown that this case contains spaces of a constant Riemann curvature of both signs.

## 1. Introduction

In recent years a new type of quantum-mechanical system has been discovered—quasi-exactly-solvable models (QESM). They possess the following characteristic property. In an infinite-dimensional space of states there exists a subspace for which eigenvalues and eigenfunctions can be found algebraically. General features of such models, methods of study and history of discovery are described in detail, from different viewpoints, in the reviews Shifman (1989), Ushveridze (1989), Ulyanov and Zaslavskii (1992). (The last review contains also a number of physical applications.)

Generally, the results obtained are related to the one-dimensional Schrödinger equation. As regards many-dimensional QESM, explicit results are obtained for the two-dimensional equation for some particular cases. In the paper by Schifman and Turbiner (1989) the Hamiltonian was taken as a quadratic–linear combination of generators of groups SU(2) × SU(2), SO(3), SU(3). In so doing, the representation of generators in terms of linear differential operators was used. The eigenvalue equation for such a Hamiltonian resulted in a differential second-order equation, which only under some conditions could be reduced to Schrödinger-like form, the corresponding space being, in general, curved.

The conditions for bringing the equation into this form represent a set of algebraic equations for coefficients of the above-mentioned combination of generators. It proves to be so cumbersome that, in the general case, it is difficult even to write down this set. To obtain quasi-exact solutions explicitly, an appropriate combination of generators should be selected. Since for different groups, generators have different structures, one should repeat the procedure.

In the present paper we develop another, more systematic, approach which generalizes the method of a previous paper for the one-dimensional case (Zaslavskii

  

1990). It is based on using, from the very beginning, the finite difference equation for two-index quantities (if it is a question of obtaining a two-dimensional Schrödinger equation). We construct the generating function which obeys the differential second-order equation as a consequence of the finite difference equation. In so doing, we lose information about the hidden dynamic algebra describing the model. However, if one is interested in the possibility of finding a new QESM in an explicit form, it is more likely to be an advantage than a disadvantage: this method enables one to achieve the purpose in the shortest way omitting all intermediate information. In addition, the method under consideration enables one to find not only particular solutions but many-parametric classes of them simultaneously. This method also admits direct generalization to many-dimensional cases when it is difficult to carry out group-theoretical analysis.

The paper is organized as follows. In section 2 we demonstrate the essence of the method using the simplest case of the one-dimensional Schrödinger equation, and derive the general equation for the generating function describing QESM. In section 3 we generalize this approach to the two-dimensional case. We study the general equation obtained for the particular class of metrics. The conditions for reducing the equation to the Schrödinger form are written down explicitly. We find the general structure of the potential and curvature. In section 4 spherically-symmetrical metrics are discussed. In section 5 we derive (without further discussion) the equation for the generating function in the $n$-dimensional case. In section 6 we sum up and formulate some problems which remain beyond the scope of the present paper.

## 2. The method of finite-dimensional matrices for the one-dimensional case

Let us consider the finite difference equation

$$\Lambda_n^m a_m = 0 \tag{1}$$

where $n$, $m$ are integers.

For reasons which become clear below, we will restrict ourselves to the case when $\Lambda_n^m$ represents a five-diagonal matrix, e.g. only $\Lambda_n^{n\pm 1}$, $\Lambda_n^{n\pm 2}$, $\Lambda_n^n$ can be non-zero. Therefore, the equation takes the form

$$\Lambda_n^n a_n + \Lambda_n^{n+1} a_{n+1} + \Lambda_n^{n-1} a_{n-1} + \Lambda_n^{n+2} a_{n+2} + \Lambda_n^{n-2} a_{n-2} = 0. \tag{2}$$

We mean that (1) and (2) are eigenvalue equations but for convenience we include the eigenvalue in the matrix element $\Lambda_n^n$. In the general case a solution of (2) represents an infinite sequence $\{a_n\}$. However, we are interested only in such solutions which are cut off, at some finite $n$, from above and below. Put the minimum value $n_{\min} = 0$. Then

$$a_n = 0 \qquad \forall n \geq N+1 \qquad n < 0. \tag{3}$$

Thus, we will consider (2) for $0 \leq n \leq N$. In so doing, 'superfluous' equations corresponding to $n = N+1$, $N+2$ and $n = -1$, $-2$ must be turned into identities. Substituting these values of $n$ into (2) and taking into account (3) we have

$$\begin{gathered} \Lambda_{N+2}^N a_N = 0 \\ \Lambda_{N+1}^N a_N + \Lambda_{N+1}^{N-1} a_{N-1} = 0 \\ \Lambda_{-2}^0 a_0 = 0 \qquad \Lambda_{-1}^0 a_0 + \Lambda_{-1}^1 a_1 = 0. \end{gathered} \tag{4}$$

These relations are valid for arbitrary $a_n$ if and only if

$$\Lambda^{N}_{N+2}=\Lambda^{N}_{N+1}=\Lambda^{N-1}_{N+1}=\Lambda^{0}_{-2}=\Lambda^{0}_{-1}=\Lambda^{1}_{-1}=0. \tag{5}$$

For matrix elements $\Lambda^m_n$ we will use second-order polynomials. This ensures that the generating function

$$\phi=\sum_{n=0}^{N} a_n x^n \tag{6}$$

obeys the second-order differential equation (each degree of $n$ acts similarly to $x(\partial/\partial x)$).

With the boundary conditions (5) taken into account, one can write down

$$\begin{aligned}
&\Lambda^n_n=\varepsilon_0+\varepsilon_1 n+\varepsilon_2 n^2\\
&\Lambda^{n+1}_n=(n+1)[\alpha_0+\alpha_1(n+1)]\\
&\Lambda^{n-1}_n=(n-N-1)[\beta_0+\beta_1(n+1)]\\
&\Lambda^{n+2}_n=\gamma(n+1)(n+2) \qquad \Lambda^{n-2}_n=\delta(n-N-1)(n-N-2).
\end{aligned} \tag{7}$$

Thus, boundary conditions do not impose any restrictions on $\Lambda^n_n$; they single out the general factor for $\Lambda^{n\pm1}_n$ and determine $\Lambda^{n\pm2}_n$ up to the constant. It is clear now why it would serve no purpose to consider equations with a more complicated structure than (2). For instance, had we taken into consideration seven-diagonal matrices, the condition of cutting off the series generalizing (5) would have led to the equality $\Lambda^{n+3}_n=0$ at three different points $n=-1, -2, -3$. For a non-zero polynomial of the second power this is impossible. On the other hand, polynomials of the third power would have given rise to a third-order differential equation for $\phi$.

Now multiply (2) with coefficients (7) by $x^n$ and carry out summation with respect to $n$. Then it turns out that the structure (7) of coefficients $\Lambda^m_n$ enables us to obtain the closed differential equation for $\phi$. For example, the term with $a_{n+1}$ gives us

$$\begin{aligned}
\sum_{n=0}^{N}(n+1)[\alpha_0+\alpha_1(n+1)]a_{n+1}x^n&=\sum_{n=-1}^{N}(\ldots)\\
&=x^{-1}\sum_{n'=0}^{N+1} n'(\alpha_0+\alpha_1 n')a_{n'}x^{n'}\\
&=\sum_{n=0}^{N}(\ldots)=\left[\alpha_0\frac{\partial}{\partial x}+\alpha_1 x^{-1}\left(x\frac{\partial}{\partial x}\right)^2\right]\phi.
\end{aligned} \tag{8}$$

In the first equation of this chain we took into account the fact that the term with $n=-1$ equals zero identically, just due to the factor $(n+1)$ which guarantees one of the conditions (5) is satisfied. In the second equality we made use of the fact that $a_{N+1}=0$.

After simple calculations along similar lines we come to the differential equation

$$\frac{d^2\phi}{dx^2}(\gamma+\alpha_1 x+\varepsilon_2 x^2+\beta_1 x^3+\delta x^4)$$

$$+\frac{d\phi}{dx}\{\alpha_0+\alpha_1+(\varepsilon_1+\varepsilon_2)x+x^2[\beta_0+\beta_1(1-N)]+2x^3\delta(1-N)\}$$

$$+\phi[\varepsilon_0-\beta_0 Nx+\delta N(N-1)x^2]=0 \tag{9}$$

(all coefficients are supposed to be real).

The equation (9) is similar to equation (11) of Turbiner (1988). For comparison one should make some re-definitions ($\delta=a_{++}$, $\beta_1=a_{+0}$, $\varepsilon_2=a_{00}+a_{+-}$, $\varepsilon_0=b_0 j+a_{00}j^2$, etc.). One can eliminate the first derivative in the standard manner. Then (9) takes the Schrödinger form with a potential which is expressed in the general case in terms of elliptic functions (Turbiner 1988, Zaslavskii 1990, Ulyanov and Zaslavskii 1992).

As follows from the procedure described, each eigenvalue of the matrix equations (2) represents also an eigenvalue of the Schrödinger equation. In general, the reverse is not true: the Schrödinger equation has an infinite set of eigenvalues which cannot be obtained in the way indicated above. This is just the reason why (9) indeed describes QESM.

Thus, in the considered approach such models are constructed starting from the finite diference equation, whose structure is particularly adapted to this purpose. The procedure is based on the following points: (i) the corresponding matrix is of three- or five-diagonal type; (ii) coefficients of the matrix are polynomials of the second degree in $n$; (iii) the conditions of cutting off the sequences $\{a_n\}$ are satisfied.

Equation (9) is the most general one which can be obtained by this approach. It can be generalized to the many-dimensional case in a direct way. In the next section we will consider the two-dimensional case.

## 3. Two-dimensional QESM: the general form of the Schrödinger equation

Consider the equation

$$\Lambda_{nm}^{n'm'}a_{n'm'}=0. \tag{10}$$

We wish to guarantee the existence of solutions which are cut off at $n=0$, $N$ and $m=0$, $M$:

$$\begin{aligned} &a_{nm}=0 \qquad \forall n\geqslant N+1 \qquad m\geqslant M+1 \\ &n<0 \qquad m<0. \end{aligned} \tag{11}$$

The matrix elements in (10) are polynomials in $n$ and $m$ of the total second power. The matrix $\Lambda_{nm}^{n'm'}$ must remove indices $n$, $m$ at no more than $\pm 2$. Taking into account

(11) and repeating the reasoning of the previous section step by step, we have

$$\begin{aligned}
&\Lambda_{n\,m}^{n\,m}=c_0+c_1n+c_2n^2+d_1m+d_2m^2+2bnm\\
&\Lambda_{n\ \ m}^{n+1\,m-1}=2f(n+1)(m-M-1)\\
&\Lambda_{n\ \ m}^{n-1\,m+1}=2g(n-N-1)(m+1)\\
&\Lambda_{n\ \ m}^{n+1\,m+1}=2e(n+1)(m+1)\\
&\Lambda_{n\ \ m}^{n-1\,m-1}=2h(n-N-1)(m-M-1)\\
&\Lambda_{n\ \ m}^{n+1\,m}=(n+1)[k+l(n+1)+2sm]\\
&\Lambda_{n\ \ m}^{n-1\,m}=(n-N-1)[\alpha(n-1)+2\beta m+\gamma]\\
&\lambda_{n\,n}^{n\,m+1}=[r+2tn+p(m+1)](m+1)\\
&\Lambda_{n\,m}^{n\,m-1}=[2\delta n+\omega(m-1)+\varepsilon](m-M-1)\\
&\Lambda_{n\ \ m}^{n+2\,m}=A(n+1)(n+2)\qquad \Lambda_{nm}^{nm+2}=C(m+1)(m+2)\\
&\Lambda_{n\ \ m}^{n-2\,m}=B(n-N-2)(n-N-1)\\
&\Lambda_{n\,m}^{n\,m-2}=D(m-M-2)(m-M-1)
\end{aligned}\tag{12}$$

(the rest of matrix elements are identically zero).

The generating function reads

$$\phi=\sum_{\substack{0\leqslant n\leqslant N\\ 0\leqslant m\leqslant M}} a_{nm}x^ny^m. \tag{13}$$

Multiply (10) by $x^ny^m$ and carry out summation with respect to $n$ and $m$. Then we obtain the closed equation for $\phi$:

$$-g^{\mu\nu}\frac{\partial^2\phi}{\partial X^\mu\partial X^\nu}+T^\mu\frac{\partial\phi}{\partial X^\mu}+V\phi=0. \tag{14}$$

Here

$$V=C_0+2hMNxy-\gamma Nx-\varepsilon My+BN(N-1)x^2+DM(M-1)y^2 \tag{15a}$$

$$\begin{aligned}
&-g^{xx}=A+lx+c_2x^2+\alpha_3x^3+Bx^4\\
&-g^{yy}=C+py+d_2y^2+\omega y^3+Dy^4\\
&-g^{xy}=e+sy+tx+gx^2+fy^2+bxy+\beta yx^2+\delta xy^2+hx^2y^2
\end{aligned}\tag{15b}$$

$$\begin{aligned}
T^x=&\ k+l+(c_1+c_2)x-2Mfy-2\delta Mxy+x^2[\alpha(1-N)+\gamma]\\
&-2hMyx^2+2B(1-N)x^3\\
T^y=&\ p+r+(d_1+d_2)y-2Ngx-2\beta Nxy+y^2[\omega(1-M)+\varepsilon]\\
&-2hNxy^2+2D(1-M)y^3.
\end{aligned}\tag{15c}$$

The equation (14) can be rewritten in terms of covariant derivatives with respect to the metric $g^{\mu\nu}$:

$$-g^{\mu\nu}(\nabla_\mu-A_\mu)(\nabla_\nu-A_\nu)\phi+U\phi=0. \tag{16}$$

Meanwhile, the Schrödinger equation in Riemannian space must take the form

$$-\Delta\Psi+U_{\rm ef}\Psi=0 \tag{17}$$

$$\Delta\Psi=\frac{(\sqrt{|g|}\,g^{\mu\nu}\Psi,\mu)_{,\nu}}{\sqrt{|g|}}.$$

Here $\Delta\Psi$ is a two-dimensional Laplacian, $g$ is the determinant, $g_{\mu\nu}$ is the metric inverse with respect to $g^{\mu\nu}$. Reduction of (16) to (17) cannot be carried out in the general case. However, it becomes possible if $A_\mu$ is a pure gradient:

$$A_\mu = \theta,\mu. \tag{18}$$

Then for the function

$$\Psi = \phi e^{-\theta} \tag{19}$$

we indeed obtain (17) by direct substitution. In so doing, comparison of coefficients at $\Psi$ and $\partial\Psi/\partial X\mu$ with the corresponding ones in (17) gives us

$$A^\mu = g^{\mu\nu}A_\nu = \frac{T^\mu + p^\mu}{2}, \qquad p^\mu = \frac{(g^{\mu\nu}\sqrt{g})_{,\nu}}{\sqrt{g}} \tag{20}$$

$$U_{\rm ef} = V + g^{\mu\nu}A_\mu A_\nu - \frac{(\sqrt{g}A^\mu)_{,\mu}}{\sqrt{g}} \tag{21}$$

For the condition (18) to be satisfied it is necessary that

$$A_{x,y} - A_{y,x} = 0. \tag{22}$$

One should calculate $A^\mu$ according to (20), find the matrix elements $g_{\mu\nu}$ inverse to (15b), compute $A_\mu$ and substitute into (22). As a result of such a substitution we obtain a fraction whose numerator represents a polynomial in $x$ and $y$. Equating its coefficients to zero, we obtain a set of algebraic equations for quantities entering (15). In the general case the corresponding system appears to be so complicated and contains so large a number of equations that it is difficult to write them down explicitly, or indeed to investigate them. This is the reason why only some separate examples of QESM were found by means of constructing special combinations of group generators with specially selected coefficients (Shifman and Turbiner, 1989).

We choose another approach. Its key idea consists in restriction to a class of metrics which enables one to find and describe a whole class of QESM explicitly. The simplest choice of a diagonal (in coordinates $x$, $y$) metric leads to a trivial system since according to (15) with $g^{xy} = 0$ variables in (14) and (17) are separated. However, the system becomes non-trivial for the metric which in our coordinate system is off-diagonal:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \frac{2dx\, dy}{a} \tag{23}$$

where $a = g^{xy}$.

Unfortunately, this metric has the signature $(1, -1)$ (note that it also takes place for models (25) and (e) in Chapter (ii) of Shifman and Turbiner 1989). Inasmuch as the case in point is the Schrödinger equation such models do not have clear physical meaning. To avoid this drawback, one can make the coordinate transformation

$$x = u + iv \qquad y = u - iv \tag{24}$$

whence the metric (23) takes the form

$$ds^2 = \frac{du^2 + dv^2}{c} \tag{25}$$

where $c = a/2$.

In general, the system remains too complicated. Therefore, we will restrict ourselves to the case when the matrix governing (10) is symmetrical in the sense that

$$\Lambda_{n\,m}^{n'm'} = \Lambda_{m\,n}^{m'n'}. \tag{26}$$

Then it immediately follows from (12) that

$$M=N \qquad c_1=d_1 \qquad f=g \qquad k=r$$
$$s=t \qquad \varepsilon=\gamma \qquad \beta=\delta. \tag{27}$$

Note also that (26) and (27) guarantee that (14) and (17) are real in variables $u$, $v$.

Direct calculation shows that the condition (22) entails the following set of equations:

$$\beta c_1 - \gamma b + 2(f\gamma - hk + fN\beta - hSN) = 0 \tag{28a}$$

$$2fc_1 - \gamma s - \beta k + 2N(fb - \beta S) = 0 \tag{28b}$$

$$sc_1 + 2N(fs - e\beta) - kb + 2(fk - e\gamma) = 0. \tag{28c}$$

Thus, we have three equations for eight quantities.

Now the coefficient $c$ in (25) is given by

$$c = e + 2su + 4fu^2 + r^2(b - 2f) + 2\beta r^2 u + hr^4 \qquad r^2 = u^2 + v^2 \tag{29}$$

while coefficients $T^\mu$ entering (14) are given by

$$\begin{aligned} T^u &= k + (c_1 - 2Nf)u + u^2(\gamma - 2N\beta) - v^2(\gamma + 2N\beta) - 2Nhur^2 \\ T^v &= v(c_1 + 2Nf + 2\gamma u - 2hNr^2). \end{aligned} \tag{30}$$

The Schrödinger equation takes the form

$$(-\tfrac{1}{2}\Delta + U_{\mathrm{ef}} - c_0)\Psi = 0 \tag{31}$$

with the potential

$$U_{\mathrm{ef}} = (2c)^{-1}(\mu_0 + 2\mu_1 u + \mu_2 r^2 + 4\kappa u^2 + \mu_3 r^2 u + \mu_4 r^4) \tag{32}$$

where coefficients are

$$\begin{aligned} \mu_0 &= k^2 - 2ks + 2ec_1 \\ \mu_1 &= 2N(fs - e\beta + e\gamma - fk) + 2(e\gamma - fk) + kc_1 + c_1 s - kb \\ \mu_2 &= 4N(N+2)(f^2 - eh) + 4N(\beta s + fc_1 - bf - \beta k) + c_1^2 + 2(s\gamma - \beta k - k\gamma) \\ \kappa &= 2N(bf - \beta s + \gamma s - fc_1) + \gamma s + \gamma k - \beta k \\ \mu_3 &= 4N(2N+3)(f\beta - hs) + 4(N+1)(f\gamma - hk) + 2(2N+1)(b\gamma - c_1\beta) + 2c_1\gamma \\ \mu_4 &= 4N(N+1)(\beta^2 - bh) + 2(2N+1)(\beta\gamma - hc_1) + \gamma^2. \end{aligned} \tag{33}$$

The curvature is

$$\begin{aligned} R &= -\frac{4}{c}(\omega_0 + \omega_1 u + \omega_2 r^2 + \tilde{\omega}_2 u^2 + \omega_3 r^2 u + \omega_4 r^4) \\ \omega_0 &= s^2 - be, \qquad \omega_1 = 4(fs - \beta e) \\ \omega_2 &= 2(\beta s - bf) + 4(f^2 - he) \\ \tilde{\omega}_2 &= 4(bf - \beta s) \\ \omega_3 &= 4(f\beta - hs) \qquad \omega_4 = \beta^2 - hb. \end{aligned} \tag{34}$$

The Schrödinger equation (31) with the potential (32) whose coefficients (33) are expressed in terms of quantities obeying three relations (28) is the main result of our paper. It represents a new class of QESM in a Riemann space with the metric (25).

The expressions obtained must be supplemented with the condition of normalizability

$$(\Psi, \Psi) = \int \mathrm{d}u\, \mathrm{d}v \sqrt{g}\, \phi^2 \mathrm{e}^{-2\theta} < \infty . \tag{35}$$

It is worth stressing that this condition in general is not satisfied automatically for solutions we deal with. One must check it every time for any model under consideration. In particular, it may turn out that it holds true only in some region of the parameters of the problem, while for other values the relationship between the spectrum of the finite-dimensional problems (10) and that of the Schrödinger equation is lost.

## 4. Spherically-symmetrical metrics and separation of variables

In this section we report the results of the analysis of general formulae of the previous section in particular cases. The simplest one of a flat space turns out to be of little interest: examination of (28) along with $R=0$ according to (34) shows that the potential represents a two-dimensional harmonic oscillator or even reduces to a constant. This means that for non-trivial QESM to be obtained in our approach the metric (25) must describe a curved space.

We will consider here the situation when the coefficient $c$ in the two-dimensional metric depends on the radius only:

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2 + r^2 \mathrm{d}\varphi^2}{c(r)} \qquad u = r\cos\varphi \qquad v = r\sin\varphi . \tag{36}$$

First, this may be of interest in physical applications and serve as a prototype for more realistic three-dimensional models. Second, since spherically symmetrical case admits separation of variables, it gives the possibility to elucidate the relation between two-dimensional and one-dimensional QESM.

First note that the choice

$$c = (1 - r^2)^2 \tag{37}$$

(which is consistent with (28)) gives us a space of constant curvature $R = -8$. That in a space with $R = \text{constant} < 0$ one can find QESM, was indicated by Shifman and Turbiner (1989). (Although the corresponding expression for the metric listed in equation (47) of their paper differs from (37) of ours, the two spaces of constant curvature are, in fact, isometric.) The model we found is more general than that of Shifman and Turbiner (1989) since their potential vanishes while ours does not. Moreover, in general in (32) $\mu_1, \mu_3 \neq 0$, so the potential, in contrast to the metric, is not spherically symmetrical.

Also, in our approach, spaces with positive constant curvature can be obtained. It is easy to check that the metric

$$c = (1 + r^2)^2 \tag{38}$$

corresponds to $R = 8$.

The manifold described by (37) is non-compact whereas the metric $c(r)$ from (38) corresponds to a compact manifold.

Let now both the metric and potential be functions of $r$ only ($\mu_1=\mu_3=0$). Then the two-dimensional Schrödinger equation (31) admits separation of variables:

$$\Psi=\chi(r)e^{im\varphi} \qquad m \text{ is integer.} \tag{39}$$

By further transformation of variables the radial part of the Schrödinger equation can be reduced to the one-dimensional Schrödinger equation. However, in general it needs very cumbersome (although direct) calculations. Therefore, we restrict ourselves to consideration of several examples of physical interest.

First, consider the case of spaces with a constant curvature. If the metric takes the form (37), substitution of

$$\begin{aligned} r&=\tanh x \\ \chi&=\tilde{\psi}\sin x\cos x \end{aligned} \tag{40}$$

leads to the Schrödinger equation

$$-\frac{d^2\tilde{\psi}}{dx^2}+\tilde{U}\tilde{\Psi}=2c_0\tilde{\Psi} \tag{41}$$

with the potential

$$\begin{aligned} \tilde{U}&=\tilde{c}_1\cosh^4 x-\tilde{c}_1[\tilde{c}_1-4(N+1)]\cosh^2 x+\left(m^2-\frac{1}{4}\right)(\sinh^{-2}x-\cosh^{-2}x)+\tilde{U}_0 \\ \tilde{c}_1&=c_1-2N \qquad \tilde{U}_0=4N+1-2\tilde{c}_1(2N+1). \end{aligned} \tag{42}$$

This is nothing but generalization of the potential of the type VIII from the table in Turbiner (1988), due to the additional third term in (42). (For comparison one needs to put in the formulae of Turbiner $\alpha=-1$, $a=2N+1$).

The case (38) can be considered in a similar way. Now the change of variables reads

$$\begin{aligned} r&=\tan x \\ \chi&=\tilde{\psi}\sinh x\cosh x. \end{aligned} \tag{43}$$

The potential in (41) proves to be

$$\begin{aligned} \tilde{U}&=-\tilde{c}_1^2\cos^4 x+\cos^2 x\,\tilde{c}_1[\tilde{c}_1+4(N+1)]+(m^2-\tfrac{1}{4})(\sin^{-2}x+\cos^{-2}x)+\tilde{U}_0 \\ \tilde{c}_1&=c_1+2N \qquad \tilde{U}_0=-2\tilde{c}_1(1+2N)-4N-1. \end{aligned} \tag{44}$$

Note that potentials (42) and (44) arise in a natural way in physical applications describing spin–phonon and spin–spin interactions (see section 3.5 of the review by Ulyanov and Zaslavskii 1992). Thus, the approach developed in the present paper turns out to cover a range of physically relevant examples.

In a similar way one can show that (28) admits the metric with the coefficient

$$c=1-r^2. \tag{45}$$

The potential is

$$\tilde{U}=\frac{c_1^2+2c_1+\frac{3}{4}}{\cos^2 x}+\frac{m^2-\frac{1}{4}}{\sin^2 x}-m^2-c_1^2 \qquad r=\sin x \qquad \tilde{\psi}=\chi(\tan x)^{1/2}. \tag{46}$$

The manifold is non-compact, the curvature $R=4/(1-r^2)$ tends to infinity when $r\to 1$.

In a similar way for

$$c=1+r^2 \tag{47}$$

we obtain (41) with the potential

$$\tilde{U}=-\frac{(c_1^2-2c_1+\frac{3}{4})}{\cosh^2 x}+\frac{(m^2-\frac{1}{4})}{\sinh^2 x}+m^2+c_1^2, \qquad r=\sinh x \qquad \tilde{\psi}=\chi(\tanh x)^{1/2}. \tag{48}$$

Thus, in both these last cases, separation of variables leads to well known exactly solvable models with Pöschl–Teller potentials.

Note also that potentials $(m^2-\frac{1}{4})$ $(\sinh^{-2}x-\cosh^{-2}x)$ and $(m^2-\frac{1}{4})$ $(\sin^{-2}x+\cos^{-2}x)$ arises for the metrics (37) and (38) if we put in (42) and (44) $\bar{c}_1=0$. This follows from direct substitution of (28), taking into account that the two-dimensional potential $U_{ef}$ (32) in this case reduces to a constant ($U_{ef}$ should not be confused with $\tilde{U}$). In other words, QESM in spaces of constant curvature with $u_{ef}=$constant (Shifman and Turbiner 1989) are, in fact, exactly solvable, being described by Pöschl–Teller potentials.

We now discuss the issue of normalizability. First consider the most interesting case (37). The factor $e^{-\theta}$ entering the expression for the wavefunction (19) can be computed directly with the help of (18)–(20) and (30):

$$e^{-\theta}=(1-r^2)^{-N}\exp\left(\frac{\nu}{r^2-1}\right) \qquad \nu=N-\frac{c_1}{2}-\gamma\cos\varphi. \tag{49}$$

As far as $\phi$ is concerned, this quantity represents a polynomial in $r$ and therefore, convergency of $(\psi,\Psi)$ (35) is determined by behaviour of the exponent at $r\to 1$ (i.e. the sign of $\nu$). If $N-(c_1/2)>\gamma$, $\Psi$ is normalizable in the region $r\leqslant 1$ (in the opposite case it is normalizable in the region $r\geqslant 1$). If $|N-(c_1/2)|\leqslant\gamma$ the quantity $(\Psi,\Psi)$ diverges.

For the metric (38) the wavefunction describing QESM appears to be normalizable independently of the values of the parameters of the problem.

For metrics (45) and (47) with a spherically-symmetrical potential, the condition of normalizability reduces simply to that for bound states of the one-dimensional Schrödinger equation (41):

$$\int dx\,\tilde{\Psi}^2<\infty. \tag{50}$$

## 5. A general $n$-dimensional case

One of the main advantages of the suggested approach is that it enables us to find the general structure of the equation which the generating function obeys. This is achieved by direct generalization of the method discussed in sections 2 and 3 for one- and two-dimensional cases. Therefore, we list the corresponding formulae without detailed discussion.

Now the matrix equation contains discrete variables $n_1$, $n_2$, ... $n_S$. It includes terms in which indices are removed no more than at $\pm 2$ for no more than two variables simultaneously. The conditions of cutting off the finite difference equation at $n_i = N_i$ for each $i = 1, 2, \ldots, S$ dictates the structure of such an equation:

$$\begin{aligned} a_{n_1\ldots n_s}(c_0 + c_i n_i + d_{ij} n_i n_j) &+ 2f_{ij}(n_i+1)(n_j - N_j - 1)a_{\ldots n_i+1\ldots n_j-1\ldots} \\ &+ e_{ij}(n_i+1)(n_j+1)a_{\ldots n_i+1\ldots n_j+1\ldots} \\ &+ h_{ij}(n_i - N_i - 1)(n_j - N_j - 1)a_{\ldots n_i-1\ldots n_j-1\ldots} \\ &+ (k^i + 2k^{ij}n_j)(n_i+1)a_{\ldots n_i+1\ldots} + (l^i + 2l^{ij}n_j)(n_i - N_i - 1)a_{\ldots n_i-1\ldots} \\ &+ A_i(n_i+1)(n_i+2)a_{\ldots n_i+2\ldots} + B_i(n_i - N_i - 2)(n_i - N_i - 1)a_{\ldots n_i-2\ldots} = 0. \end{aligned} \tag{51}$$

Here summation is taken with respect to repeating indices

$$e_{ij} = e_{ji} \qquad h_{ij} = h_{ji} \qquad f_{ii} = e_{ii} = h_{ii} = 0.$$

Multiply (51) by $x_1^{n_1}, \ldots, x_s^{n_s}$ and sum over all values of $n_1, \ldots, n_S$. Then for the generating function

$$\phi = \sum_{n_1,\ldots n_s} a_{n_1\ldots n_s} x_1^{n_1} \ldots x_s^{n_s} \tag{52}$$

one can obtain the closed differential second-order equation

$$-g^{ij}\frac{\partial^2\phi}{\partial x^i \partial x^j} + T^i \frac{\partial \phi}{\partial x^i} + V\phi = 0 \tag{53}$$

where the coefficients are given by (there is no summation over $i$ and $j$ here)

$$-g^{ij} = A_i\delta_{ij} + e_{ij} + k^{ij}x_j + k^{ji}x_i + d_{ij}x_i x_j + f_{ij}x_j^2 + f_{ji}x_i^2 \tag{54a}$$

$$\begin{aligned} T^i = k^i + (c_i + d_i)x_i - 2x_i \sum_e f_{ie}N_e - 2x_i^2 N_i \sum_e h_{ie}x_e \\ + x_i^2(e^i + 2e^{ii}) - 2x_i \sum_p N_p x_p l^{pi} + 2B_i x_{3i}(1 - N_i) \end{aligned} \tag{54b}$$

$$V = c_0 + 2\sum_{m<n} h_{mn}N_m N_n x_m x_n - \sum_m N_m(l^m + 2l^{mm}x_m) + \sum_m B_m N_m(N_m - 1)x_m^2. \tag{54c}$$

The Schrödinger equation can be obtained by means of substitution (19) but only if the conditions of integrability which generalize (22) are satisfied. We will not study here the properties of (53) and (54) since it is a theme for a separate investigation. It is only worth noting that (54) gives us, in our framework, the most general structure of the equation for $\phi$ describing QESM in the class of polynomials in $x_i$. Note that QESM which can be obtained from the known ones in a flat space by means of Darboux, Gelfand–Levitan or other transformations (Ushveridze 1991) are unrelated to the subject of our paper.

## 6. Summary and outlook

We have suggested a simple and effective method which enables us to formulate the equation for the generating function of quasi-exact solutions. In so doing, we need not either derive expressions for group generators as linear differential operators (see appendix A of the review by Shifman 1989) or select special combinations of them, as were made by Shifman and Turbiner (1989). Instead, we deal with finite-dimensional matrices. It is of interest to compare the two approaches directly and elucidate whether they are equivalent. In addition, the following question arises: can a quasi-exactly-solvable system with a given finite matrix be described in terms of more than one dynamic algebra?

Since there is no oscillation theorem for many-dimensional Schrödinger equation one cannot state that solutions obtained belong certainly to the low-lying part of the spectrum, as is the case for one-dimensional systems (Ulyanov and Zaslavskii 1992). It is of interest to elaborate classification of energy levels corresponding to quasi-exact solutions.

With the help of the method suggested we have managed to find the many-parametric class of two-dimensional models. It is especially interesting that this class contains spaces with a constant curvature. Generalization of this result to the three-dimensional case could be of interest for relativistic cosmology.

Additionally, it is worth investigating two-dimensional models with a metric more general than that considered in the present paper.

Finally, it is interesting to elucidate whether it is possible to obtain QESM with a magnetic field.

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